NON-ABELIAN POINCARE DUALITY AND DRINFELD'S PROOF OF CONTRACTIBILITY OF RATIONAL MAPS

NIKOLAY GRANTCHAROV

Abstract. These expository notes accompany my BunG seminar talk on non-abelian Poincare duality. The goal of this note is twofold. First, we describe the main ingredients in the proof of said topic, following Lurie's lecture notes. Then we elaborate on the main step – the space of rational maps from a smooth projective curve X to a semisimple group G is cohomologically contractible. We give a simple and new proof of this remarkable fact due to Drinfeld.

1. Non-abelian Poincare duality

Recall our setup: Let k be an algebraically closed field, X a smooth projective curve over k, and G a smooth affine group scheme over X. We will assume G has a semisimple and simply connected generic fiber over X. The goal this quarter is to compute the mass of $\mathrm{Bun}_G(X)$, i.e to prove the Gaitsgory-Lurie Tamagawa number conjecture. This requires us to understand the l-adic (co)homology of the stack $\mathrm{Bun}_G(X)$. The main goal of this note is to give some explicit realization of $H_*(\mathrm{Bun}_G(X), \mathbf{Z}_l)$ using non-abelian Poincare duality, which we now explain.

Consider the Beilinson-Drinfeld Grassmannian $\mathrm{Gr}_{G,Ran}$ which parameterizes principal G-bundles on X, together with a trivialization away from finitely many points of X. More precisely:

Definition 1.1. *Define the prestack $Gr_{G,Ran}$ whose S points, for S some test affine k-scheme, consist of principal G-bundles P over $S \times X$, together with a finite non-empty set of maps $f_1, \dots, f_n : S \to X$, and together with a trivialization of P on $S \times X \setminus \Gamma_{f_1} \cup \cdots \cup \Gamma_{f_n}$, where Γ_{f_i} are the graphs of f_i.*

Then $\mathrm{Gr}_{G,Ran}$ comes equipped with two forgetful maps:

$$\mathrm{Ran}(X) \xleftarrow{\ \phi\ } \mathrm{Gr}_{G,Ran} \xrightarrow{\ \rho\ } \mathrm{Bun}_G(X)$$

Here, $\mathrm{Ran}(X) = \mathrm{Gr}_{e,Ran}$ (So $G = e$ is the trivial group) is the prestack whose S points are finite nonempty subsets of the S points of X. The main theorem states:

Theorem 1.2 (Non-abelian Poincare duality)**.** *Suppose the generic fiber of G is semisimple and simply connected. Suppose $l^{-1} \in k$. Then the forgetful map $\rho : Gr_{G,Ran} \to Bun_G(X)$ induces an isomorphism on l-adic homology and cohomology:*

$$H_*(Gr_{G,Ran}, \mathbf{Z}_l) \xrightarrow{\sim} H_*(Bun_G(X), \mathbf{Z}_l).$$

Date: February 14, 2024.

Drinfeld's proof was outlined by Gaitsgory at the Geometric Langlands office hours in 2023 [G2], and these notes stemmed from an attempt to elaborate on those lectures and make explicit each step. I'd like to give special thanks to Justin Campbell and Kevin Lin for explaining many steps in the proof and for significantly improving this document!

1.1. **l-adic homology for prestacks.** Lets take a brief detour to recall how we define the l-adic homology for prestacks [G1, Section 1.6]. Suppose $\mathcal{Y}$ is a prestack which may be written as $\mathcal{Y} = \mathrm{colim}_{I\in\mathcal{S}} Y_I$ where Y_I are schemes and $\mathcal{S}$ is an arbitrary category of indices. If we further require transition maps to be closed embeddings (resp. proper), then we call such a prestack $\mathcal{Y}$ an indscheme (resp. pseudo-indscheme). Then we can define the category of sheaves on $\mathcal{Y}$ as

$$\mathrm{Shv}^!(\mathcal{Y}) := \lim_I \mathrm{Shv}(Y_I)$$

where the limit is taken with respect to !-pullback and in the category of stable ∞-categories. Then given a category of sheaves on $\mathcal{Y}$, we can take homology to be defined as

$$H_*(\mathcal{Y}) := \Gamma_c(\mathcal{Y}, \omega_{\mathcal{Y}}) = \mathrm{colim}_I \Gamma_c(Y_I, \omega_{Y_I}),$$

where $\omega_{\mathcal{Y}}, \omega_{Y_I}$ are the canonically defined dualizing sheaves on the respective spaces, and

$$\Gamma_c(Y_I, \omega_{Y_I}) := (p_{Y_I})_! \omega_{Y_I}, \quad \text{where } p_{Y_I} : Y_I \to \mathrm{pt}$$

is the homology with compact support. Given $f : S \to T$ a morphism of affine schemes, we have $f^!\omega_T = \omega_S$. Hence, the induced transition maps are obtained by applying Γ_c to the counit morphism: $f^! f_! \omega_T \to \omega_T$. We may summarize the above construction using (even more) sophisticated language: Suppose we have a homology theory from affine schemes to vector spaces. Then Yoneda provides an embedding of affine schemes to prestacks, and the process of extending this homology theory to prestacks is given by "Left Kan extension".

$$\begin{array}{ccc} \mathrm{Sch}^{aff} & \xrightarrow{H_*(-)} & \mathrm{Vect} \\ \big\downarrow & \nearrow & \\ \mathrm{PreStack} & & \end{array}$$

This formalism allows us to define l-adic homology for all prestacks of interest, i.e $\mathrm{Bun}_G(X)$, $\mathrm{Gr}_{G,\mathrm{Ran}}, \mathrm{Maps}_{Ran}(X,Y), \mathrm{Maps}_{gen}(X,Y)$. This also highlights an advantage to using the Ran-version for rational maps and the affine Grassmannian – they are automatically (pseudo-)indschemes and D-module theory and l-adic homology is more manageable for pseudo ind-schemes.

1.2. **!-fibers of $H_*(\mathrm{Bun}_G, \omega_{\mathrm{Bun}_G})$.** Note, we may write

$$H_*(\mathrm{Bun}_G, \omega_{\mathrm{Bun}_G}) = H_*(\mathrm{Gr}_{G,Ran}, \omega_{\mathrm{Gr}_{G,Ran}}) = H_*(\mathrm{Ran}(X), \phi_!(\omega_{\mathrm{Gr}_{G,Ran}}))$$

where the first equivalence is Theorem 1.2 and second one is definition. The latter interpretation has the advantage that the $\mathrm{Ran}(X)$ space is independent of the group scheme G, and $\phi_!(\omega_{\mathrm{Gr}_{G,Ran}})$ only depends on the local behavior of G! Furthermore, it is easy to explain what happens along fibers of ϕ. Namely, $\phi^{-1}(x)$ consists of G-bundles on X equipped with a trivialization on $X \setminus \{x\}$. Using the uniformization theorem for $\mathrm{Bun}_G(X)$, we see this is just the affine Grassmannian $\mathrm{Gr}_G^x := G(K_x)/G(O_x)$, where $O_x \simeq k[[t_x]], K_x \simeq k((t_x))$. This is an ind-projective variety, and $\phi : \mathrm{Gr}_{G,Ran} \to \mathrm{Ran}(X)$ is ind-proper. Thus $\phi_!(\omega_{\mathrm{Gr}_{G,Ran}}) = \phi_*(\omega_{\mathrm{Gr}_{G,Ran}})$, and therefore

$$i_x^! \phi_!(\omega_{Gr_{G,Ran}}) = \Gamma_c(\mathrm{Gr}_G^x, \omega_{Gr_G^x}).$$

There is a natural generalization: take a finite set $I = \{x_1, \ldots, x_n\} \in \mathrm{Ran}(X)$ and let $i_I : I \hookrightarrow \mathrm{Ran}(X)$ be the inclusion. Then $\phi^{-1}(I) = \mathrm{Gr}_G^{x_1} \times \cdots \times \mathrm{Gr}_G^{x_n}$ and

$$i_I^! \phi_!(\omega_{Gr_{G,Ran}}) = \bigotimes_{i\in I} \Gamma_c(\mathrm{Gr}_G^{x_i}, \omega_{Gr_G^{x_i}}).$$

1.3. **Sketch of proof for non-abelian Poincare duality.** Let us now sketch the proof of non-abelian Poincare duality. First, to show that $\mathrm{Gr}_{G,Ran} \to \mathrm{Bun}_G(X)$ induces an isomorphism on l-adic homology, it suffices to check it locally. Namely we'd like to show for any $S := \mathrm{Spec}(R)$-point of $\mathrm{Bun}_G(X)$ and corresponding Cartesian diagram,

$$\begin{array}{ccc} S \times_{\mathrm{Bun}_G(X)} \mathrm{Gr}_{G,Ran} & \longrightarrow & \mathrm{Gr}_{G,Ran} \\ \downarrow{\scriptstyle \pi_{\mathcal{P}}} & & \downarrow{\scriptstyle \rho} \\ S & \xrightarrow[\mathcal{P}]{} & \mathrm{Bun}_G(X) \end{array}$$

the projection map $\pi_{\mathcal{P}}$ induces an isomorphism in l-adic homology. Denote

$$\mathrm{Sect}(\mathcal{P}) := S \times_{\mathrm{Bun}_G(X)} \mathrm{Gr}_{G,Ran}.$$

Let us now introduce two prestacks of rational maps:

Definition 1.3. *Let X be a smooth, connected projective curve and Y an affine scheme. Define the prestack*

$$Maps_{Ran}(X, Y) : (Sch^{aff})^{op} \to Set \subset \infty\text{-}Grpd$$

whose value on affine test schemes S consist of the data of a nonempty finite subset $\underline{x} \subset Maps(S, X)$ plus a rational map $S \times X \to Y$ which is regular on $S \times X \setminus \Gamma_{\underline{x}}$.

We may express $\mathrm{Map}_{Ran}(X, Y)$ as a colimit of certain ind-schemes $\mathrm{Maps}_{X^I}(X, Y)$.

Definition 1.4. *Let X be a smooth, connected projective curve and Y an affine scheme. Define the prestack*

$$Maps_{gen}(X, Y) : (Sch^{aff})^{op} \to Set \subset \infty\text{-}Grpd$$

whose value on affine test schemes S consist of the collection of maps $m : U \to Y$ where U ranges over the open subsets of $S \times X$ such that $\forall s \in S$, $U \cap (s \times X) \neq \emptyset$, and two maps $m_1 : U_1 \to Y$, $m_2 : U_2 \to Y$ are identified if they are equal on the intersection $U_1 \cap U_2$.

The main difference between the Ran and generic prestack of rational maps is that Ran specifies the location of the poles whereas $\mathrm{Maps}_{gen}(X, Y)$ does not. Hence, there is a natural forgetful map of prestacks

$$\mathrm{Map}_{Ran}(X, Y) \to \mathrm{Map}_{gen}(X, Y).$$

In fact, this map induces an isomorphism on l-adic homology [G1, Cor. 2.3.6]. A heuristic explanation for this is that the fibers of this map are acted on transitively by $\mathrm{Ran}(X)$ with its semigroup structure, and $\mathrm{Ran}(X)$ is homologically contractible (See Section 2.1 below).

Let us explain a series of steps to reduce Theorem 1.2 to the claim that the space of rational maps from X to G is contractible.

(1) If $\mathcal{P}$ and $\mathcal{P}'$ are G-bundles on $S \times X$ which are isomorphic over an open set $S \times X \setminus \Gamma_{\underline{x}}$ for some finite subset $\underline{x} \subset Maps(S, X)$, then the projection $Sect(\mathcal{P}) \to S$ induces an isomorphism on l-adic homology if and only if $Sect(\mathcal{P}') \to S$ does.

Indeed, this follows because $\mathrm{Sect}(\mathcal{P})$ is just identified with the $\mathrm{Spec}(R)$ points of $\mathrm{Sect}_{Ran}(X, \mathcal{P})$, and this space is homologically equivalent to $\mathrm{Sect}_{gen}(X, \mathcal{P})$ of generic sections $X \to \mathcal{P}$, and the latter is insensitive to replacing X by some Zariski open $U \subset X$ because it only captures the generic behavior of maps.

(2) Every principal G-bundle $\mathcal{P}$ over $S \times X$ is generically trivial (after passing to some etale cover). Hence we may assume $\mathcal{P} = \mathcal{P}_{triv}$ is the trivial G bundle on $S \times X$.

We say $\mathcal{P}$ is "generically trivial" over $S \times X$ if there is some open $U \subset S \times X$ for which $U \to S$ is surjective and $\mathcal{P}$ is trivial on U. Existence of a generic trivialization (on an etale cover of S, but this is insensitive to homology) follows from Drinfeld and Simpson (and discussed in Minh-Tam's talk) in the case G is split. In Lurie lecture 13, it is explained how to modify Drinfeld-Simpson theorem for the non-split case. Thus we can just take U to be the open for which $\mathcal{P}$ trivializes, and consequently may just assume $\mathcal{P}$ is trivial.

(3) We may assume G is generically split and $R = k$. In this case, $Sect(\mathcal{P}_{triv}) = Maps_{Ran}(X, G)$ where G is now thought of as a simply-connected, semisimple algebraic group over k.

Thus, it just remains to prove the following theorem:

Theorem 1.5. *Suppose G is simply-connected, semisimple algebraic group over k, and X is a smooth projective curve over k. Then the projection $Maps_{Ran}(X, G) \to Spec(k)$ induces an isomorphism in the l-adic homology.*

The entire next section is devoted to proving Theorem 1.5

2. Proof of contractibility

2.1. **Contractibility of Ran space.** First, suppose $G = e$ is the trivial group. Then Theorem 1.5 claims the Ran space Ran$X$ is homologically trivial. This celebrated theorem is originally due to [BD, Prop 3.4.1]. We sketch the proof as done in Lurie's lecture 10. We wish to show

$$H_*(\mathrm{Ran}(X), \mathbf{Z}_l) = \begin{cases} \mathbf{Z}_l & \text{if } * = 0 \\ 0 & \text{else.} \end{cases}$$

Since X is connected, each X^I is connected, hence $H_0(X^I, \mathbf{Z}_l) = \mathbf{Z}_l$. Since $\mathrm{Ran}(X)$ is a pseudo-indscheme, its zeroth homology is the direct limit (in the ∞-category $\mathrm{Mod}_{\mathbf{Z}_l}$) of these trivial homology groups, hence is trivial. Next, suppose by induction $H_i(\mathrm{Ran}(X), \mathbf{Z}_l) = 0$ for $0 < i < n$. Let $V = H_n(\mathrm{Ran}(X), \mathbf{Z}_l)$. Then Kunneth theorem says, since all homologies for $i < n$ vanish,

$$H_n(\mathrm{Ran}(X) \times_k \mathrm{Ran}(X), \mathbf{Z}_l) = V \oplus V$$

Next, an object of $\mathrm{Ran}(X)$ is (R, S) for R a k-algebra and S a finite non-empty subset of $X(R)$. Thus there is the evident "multiplication" map given by taking union:

$$m : \mathrm{Ran}(X) \times \mathrm{Ran}(X) \to \mathrm{Ran}(X), \quad ((R, S), (R, S')) \mapsto (R, S \cup S').$$

On homology, this becomes a map $V \oplus V \to V$, which we identify with a pair of linear maps $\lambda, \mu : V \to V$. By symmetry, we must have $\lambda = \mu$. The key property that Ran$X$ satisfies is that the composition

$$\mathrm{Ran}(X) \xrightarrow{\Delta} \mathrm{Ran}(X) \times \mathrm{Ran}(X) \xrightarrow{m} \mathrm{Ran}(X)$$

is the identity, where Δ is the diagonal inclusion. Thus, $2\lambda(v) = v$ for all $v \in V$. Now, pick a point $x \in X$ and denote $\{x\} \in \mathrm{Ran}(X)$ the corresponding k-point: $i_x : \mathrm{Spec}k \to \mathrm{Ran}(X)$. Let F denote the composite map

$$F : \mathrm{Ran}(X) \xrightarrow{(i_x, \mathrm{id})} \mathrm{Ran}(X) \times \mathrm{Ran}(X) \xrightarrow{m} \mathrm{Ran}(X).$$

Passing to homology, we find that F is idempotent: $F^2 = F : V \to V$ and it is given by $v \mapsto \lambda(v)$. Thus, $2\lambda(v) = 2\lambda(\lambda(v)) = \lambda(v)$ implies $\lambda(v) = 0$ for all v. Consequently, $v = 2\lambda(v) = 0$ for all v, which implies $V = 0$. This completes the proof.

2.2. **Contractibility of space of rational maps.** Now consider the prestack of generic rational maps $\mathrm{Maps}_{gen}(X, Y)$ (See Definition 1.4), where X is a smooth projective curve and Y is a quasi-projective k-scheme. We wish to show it is cohomologically trivial when $Y = G$. Let us make two reductions:

(1) Suppose Y has a decomposition $Y = \cup_i Y_i$ such that the contractibility statement holds for all finite intersections $Y_{i_1} \cap \cdots \cap Y_{i_r}$. Then the contractibility statement holds for Y. As a corollary, we may assume $Y = \mathbf{A}^n \setminus V(f)$ is a principal open affine, where $f \in k[x_1, \ldots, x_n]$.

This follows formally because the natural map $\mathrm{colim}_I \mathrm{Maps}_{gen}(X, Y_I) \to \mathrm{Maps}_{gen}(X, Y)$ becomes an isomorphism of stacks once we take stackifications of both sides. Here, $Y_I := \cap_{i \in I} Y_i$. Then, taking l-adic homology is insensitive to stackification, so we deduce the claim. For a more rigorous proof, we direct the reader to [L, Lecture 16, Prop. 7].

(2) Reduction (1) applies to $Y = G$, a semi-simple simply-connected group over k.

Indeed, let B be a Borel subgroup, and let B^{op} be the opposite Borel so that $B \cap B^{op} = T$ is a maximal torus. Let U, U^{op} be the unipotent radical of B, resp. B^{op}. The Bruhat decomposition implies $V := U \times T \times U^{op} \subset G$ is an open, dense subset. Observe that U, U^{op} are unipotent, hence affine, and the torus T is of the form $\mathbf{A}^n \setminus V(f)$, where $f = x_1 \cdots x_n$. Since G is quasi-compact, we may write $G = \bigcup_{i \leq n} g_i V$ for some translates $g_i \in G(k)$. Note that $V_I := \cap_{i \in I} g_i V$ is an open subset of V. Hence, we may write $V_I \hookrightarrow \mathbf{A}^d$ where $d = \dim(G)$ and assume it is the complement of the vanishing of some $f \in k[x_1, \ldots, x_d]$.

Remark 1. *There exists a 3rd reduction: we may even further replace X with $\mathbf{P}^1$, or even $\mathbf{A}^1$, but we do not need it. To explain this reduction, we just consider a finite map $X \to \mathbf{P}^1$, and then use Weil restriction to conclude*

$$Maps_{gen}(\mathbf{P}^1, Res^X_{\mathbf{P}^1} Y) = Maps_{gen}(X, Y))$$

Now, when Y is affine space, its Weil restriction is affine and we're done. If $Y = \mathbf{A}^n - V(f)$ is basic open affine, then its Weil restriction is "generically" basic open affine, and generic maps only capture generic behavior, so we are done again. To go from $\mathbf{P}^1$ to $\mathbf{A}^1$, we use that replacing the source by dense open does not affect the generic maps.

In summary, we are reduced to proving the following concrete statement:

Theorem 2.1 (Drinfeld)**.** *The prestack $Map_{Ran}(X, \mathbf{A}^n - V(f))$ is acyclic in the l-adic homology.*

Proof. Let $U = \mathbf{A}^n - V(f)$. Let us first introduce the intermediate prestack

$$\mathrm{Maps}_{Ran}(X, U) \subset \mathrm{Maps}_{Ran}(X, U \subset \mathbf{A}^n) \subset \mathrm{Maps}_{Ran}(X, \mathbf{A}^n)$$

which consists of rational maps $X \to \mathbf{A}^n$ which generically land in U. This induces the same l-adic homology as $\mathrm{Maps}_{Ran}(X, U)$, as explained in [L, Lec 16, Prop. 6]

We will show the fibers of $\mathrm{Map}_{Ran}(X, U \subset \mathbf{A}^n)$ over $\mathrm{Ran}(X)$ are filtered colimits of prestacks that are $\mathbf{A}^1$-contractible. Then using that $\mathrm{Ran}(X)$ is also contractible, we will have our result. Now, fix a point $\underline{x} = (x_1, \ldots, x_r) \in \mathrm{Ran}(X)$. This induces the fiber $\mathrm{Maps}(X - \underline{x}, U \subset \mathbf{A}^n)$, consisting of regular maps $X - \underline{x} \to \mathbf{A}^n$ which generically land in U.

Consider

$$\mathrm{Maps}(X - \underline{x}, \mathbf{A}^n)_{\leq N} \subset \mathrm{Maps}(X - \underline{x}, \mathbf{A}^n),$$

the prestack of rational maps $(u_1, \ldots, u_n) : X \to \mathbf{A}^n$ such that $\deg(u_i) \leq N$ for all i, where **degree** means order of pole at x_1. We will show for any $N \geq 0$, the embedding

$$\mathrm{Maps}(X - \underline{x}, \mathbf{A}^n)_{\leq N} \cap \mathrm{Maps}(X - \underline{x}, U \subset \mathbf{A}^n) \hookrightarrow \mathrm{Maps}(X - \underline{x}, U \subset \mathbf{A}^n)$$

is homotopic to the constant map.

Using Noether normalization, we may assume f is monic in z_1 and write

$$f = z_1^d + \sum_{d' < d} z_1^{d'} f_{d'}(z_2, \ldots, z_n) \in k[z_1, \ldots, z_n]$$

Lemma 2.2. *Suppose $(u_1, \ldots, u_n) \in Maps(X - \underline{x}, \mathbf{A}^n)$ are such that $deg(u_i) \leq N$ for $i \geq 2$ and $f(u_1, \ldots, u_n) = 0$. Then there exists N', depending only on N and degree of coefficients, $deg(f_{d'})$, of f, such that $deg(u_1) \leq N'$.*

Proof of lemma. If $f(u_1, \ldots, u_n) = 0$, then $\deg(u_1^d) = \deg(\sum_{i=2}^d f_i(u_2, \ldots, u_n) u_1^i)$. If $\deg(u_i) \leq N$ for $i > 1$, then $\deg(f_i(u_2, \ldots, u_n)) \leq N'$ for some N' depending on N and coefficients of f. Then, writing u_1 as quotient of two polynomials, we find the right hand side of $\deg(u_1^d)$ has degree $\deg(u_1^d) \leq (d-1)\deg(u_1) + N'$. Thus $\deg(u_1) \leq N'$. □

Now, consider the homotopy

$$(u_1, \ldots, u_n) \mapsto (1-t)(u_1, \ldots, u_n) + (tw_1, 0, \ldots, 0)$$

where $(u_1, \ldots, u_n) \in \text{Maps}(X - \underline{x}, \mathbf{A}^n)_{\leq N} \cap \text{Maps}(X - \underline{x}, U \subset \mathbf{A}^n)$ and w_1 is chosen to be some rational function on X with order of pole at x_1 greater than N', and N' is chosen to be as in Lemma 2.2. Then, by construction, for each t, the image of this homotopy lands in $\text{Maps}(X - \underline{x}, U \subset \mathbf{A}^n)$. And at $t = 1$, we just get the constant map. $\square$

References

[BD] A. Beilinson, V. Drinfeld, Chiral Algebras
[L] J. Lurie, Lecture notes 10, 12, 13, 16, 17 on Tamagawa number
[G1] D. Gaitsgory, Contractibility of the space of rational maps
[G2] D. Gaitsgory, Lectures Jan 17, 2023 and Jan 24, 2023 at Geometric Langlands office hours